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Inverse functional relations for lattice models

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Abstract. Inverse functional equations are studied for the partition functions and for the correlation functions of various models. The validity of the inverse relation for the partition function is justified with the help of approximations by quasi-one-dimensional models defined on strips of increasing size. The possibility of determining the partition function through the use of the inverse and other symmetry relations, coupled to analyticity hypotheses, is briefly investigated on the two-dimensional Ising model with a field. The group generated by the inverse relation and the spatial symmetries of the model is studied in a three-dimensional case. Inverse relations are also exhibited, firstly for two-point and then for n-point correlation functions. They are first put into evidence by a geometric approach and then verified on a particular high-temperature expansion.

1. Introduction

The inverse relation method has been used as a quick and efficient way to find the partition functions of some two-dimensional exactly solvable models (Baxter 1980, 1982). The interest in such a method would be increased if it were possible to apply it generally, that is to say to non-solvable two-dimensional models and to three-dimensional models, or to other physical quantities such as the correlation functions. For this purpose we extend an approach which was initiated for the anisotropic two-dimensional Potts model (Jaekel and Maillard 1982b). In this approach the inverse relation is associated to the spatial symmetries of the model in order to generate an infinite discrete group and corresponding automorphic functional equations on the partition function. The method then exploits these constraints which can be seen to be intermediate between the ones coming from the commutativity of a continuous family of transfer matrices (see Kasteleyn 1975 for a review) and the one coming from an involution such as the self-duality (Kramers and Wannier 1941).

Section 2 of this paper only deals with partition functions. Taking the thermodynamic limit, it is not obvious that the inverse relation on the transfer matrix will imply the corresponding automorphic functional relation for the partition function. Firstly we indicate how to corroborate this correspondence by considering a sequence of models defined on strips of increasing size. Then, the inverse functional relation is not sufficient by itself to determine the partition function. One needs information

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on the analytical behaviour of the function. Secondly we study the implications of the inverse relation and of the constraints coming from the theorem of Lee and Yang (1952), for the two-dimensional Ising model with a magnetic field. The three-dimensional models have more spatial symmetries, thus leading to a larger automorphy group. Section 2.3 is devoted to the study of the structure of this group in the case of the three-dimensional Potts model.

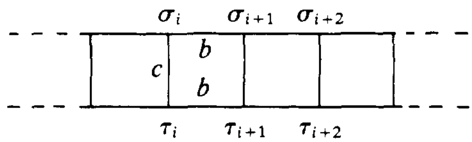
In § 3, we show how to apply the same geometric and diagrammatic methods to derive the inverse functional relation for the correlation functions of two- and three-dimensional models. Two-point as well as n-point correlation functions are studied. Special attention is given to the correlation functions having an extra geometrical symmetry, that is to say, the diagonal ones.

2. Partition functions

2.1. Quasi-one-dimensional models

The validity of the inverse relation for the partition function of models with a dimension greater than two can be shown in an instructive way by its study on strips of increasing size, which realise, as will be seen, an interpolation between dimension one and the dimension considered.

We shall study here the q-state Potts model on strips of size N, which enables one, by taking appropriate limits, to recover the percolation problem ($q \to 1$), the Ising model ($q \to 2$), or the chromatic polynomials ($b \to 0$). For instance, for the strip of size 2:

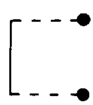


the partition function per site Z is defined by

$$Z^{2M}(b, c) = \sum_{\{\sigma,\tau\}} \prod_i b^{\delta_{\sigma_i\sigma_{i+1}}+\delta_{\tau_i\tau_{i+1}}} c^{\delta_{\sigma_i\tau_i}}, \qquad \sigma, \tau \in \mathbb{Z}q.$$

There exist two formalisms which give the partition function as the largest eigenvalue of a transfer matrix defined on the strip of size N: the most natural one leads to a transfer matrix which operates on the configurations of the spins in a column; the second one, which was introduced by Blöte *et al* (1981), considers a transfer matrix whose size is independent of q, and which operates on states that are defined by connectivity properties: for example, for $N = 2$, there are two states ϕ_1 (resp ϕ_2) corresponding to the following two configurations:

2 disconnected points: ϕ_1 2 connected points: ϕ_2



In both cases, the transfer matrix is the product of a transfer matrix T_1, which corresponds to adding the horizontal bonds of a column, and of a transfer matrix T_2, which corresponds to adding the vertical bonds of a column. In the second formalism T_1 and T_2 are constructed according to the following rule: every supplementary bond introduces a factor $(b-1)/q$ (resp $(c-1)/q$) and every new loop introduces a factor q. For instance, for $N=2$, T_1 and T_2 will have the following actions on ϕ_2

$$T_1:\quad \phi_2 \to [1+2(b-1)/q]\phi_1 + [(b-1)/q]^2\phi_2,$$

$$T_2:\quad \phi_2 \to (1+[(c-1)/q]q)\phi_2,$$

leading to the following transfer matrices:

$$T_1 = \begin{vmatrix} [1+(b-1)/q]^2 & 1+2(b-1)/q \\ 0 & [(b-1)/q]^2 \end{vmatrix}, \qquad T_2 = \begin{vmatrix} 1 & 0 \\ (c-1)/q & c \end{vmatrix}.$$

These matrices can be seen to satisfy the inverse relations

$$T_1(2-q-b) = \left(\frac{b-1}{q}\right)^2 \left(\frac{b+q-1}{q}\right)^2 T_1^{-1}(b), \qquad T_2\left(\frac{1}{c}\right) = T_2^{-1}(c).$$

Considering the transfer matrix

$$T = T_2^{1/2} T_1 T_2^{1/2}$$

one can then verify the inverse matricial equation in both formalisms:

$$T(b,c)T(2-q-b, 1/c) = [(b-1)/q]^2[(b+q-1)/q]^2 \mathbb{1}.$$

As can be seen easily for the smallest sizes of the strip ($N=2, 3, \ldots$), the transfer matrix then shows the following property: the lowest eigenvalue at the inverse point $(2-q-b, 1/c)$ is the analytic continuation to this point of the largest eigenvalue. For instance, in the simplest case ($N=2$), the eigenvalues can be seen to be the two different determinations of a binomial equation, with rational coefficients in the parameters of the model. Hence the inverse relations imply an inverse functional equation on the largest eigenvalue, that is to say the partition function per site for the strip of size N, and its analytic continuation:

$$Z(b,c)Z(2-q-b, 1/c) = (b-1)(1-q-b).$$

(Let us remark that the preceding result also holds for other problems which are defined on strips, such as the directed percolation.) It is then possible to use the partially resummed expansion in $1/b^*$, $1/c^*$ (where $b^* = (b+q-1)/(b-1)$, $c^* = (c+q-1)/(c-1)$), which was considered in Jaekel and Maillard (1982a, b) for the partition function, and to introduce it into the inverse relation. For instance in the $N=2$ case one has

$$\ln Z = \ln \frac{(b+q-1)}{q^{1/2}} \frac{(c+q-1)}{q^{1/2}} + (q-1)\frac{1}{c^{*2}}\frac{1}{b^{*2}-1} + (q-1)(q-2)\frac{1}{c^{*3}}\left(\frac{1}{b^{*2}-1}\right)^2 + \ldots$$

which satisfies the inverse relation up to the second order. This partially resummed expansion corresponds to an expansion around the one-dimensional model and it must be noticed that one can give it a diagrammatic representation so that, for a strip of size N, it coincides, up to the Nth order, with the corresponding expansion of the two-dimensional partition function (the strip of infinite size). This implies that the latter must satisfy the functional equation up to any order N.

The same considerations hold whatever the dimension is, the strips being replaced in general by a sequence of quasi-one-dimensional models whose size in the other directions increases up to infinity. Hence the partition functions of the strips build a hierarchy of solutions of the inverse functional equation, whose limit also satisfies the equation. However, this limit possesses an extra symmetry which distinguishes the partition function of the infinite lattice from those of the strips: the spatial symmetry between the different directions of the lattice, which is only restored in the infinite limit.

2.2. *The two-dimensional Ising model with a field*

Following Baxter (1980), one can envisage the inverse and symmetry relations for a model which is not exactly soluble, such as the anisotropic two-dimensional square Ising model with a magnetic field:

$$Z^{NM}(K_1, K_2, H) = \sum_{\{\sigma\}} \prod_{\langle ij\,\mathrm{hor}\rangle} \exp(K_1\sigma_i\sigma_j) \prod_{\langle kl\,\mathrm{ver}\rangle} \exp(K_2\sigma_k\sigma_l) \prod_i \exp(H\sigma_i), \qquad \sigma \in \mathbb{Z}_2.$$

This model has the following transfer matrices (see figure 1)

$$T_1 = \prod_i \exp(K_1\sigma_i\sigma_i'), \qquad T_2 = \prod_i \exp(K_2\sigma_i\sigma_{i+1} + H\sigma_i)\delta_{\sigma_i\sigma_i'}.$$

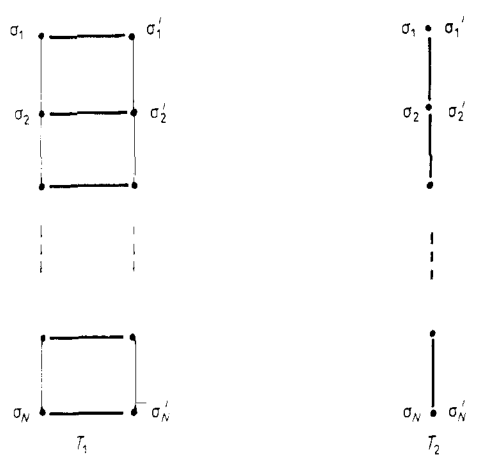


Figure 1.

Indeed, the presence of a magnetic field does not change the inverse relation:

$$T_1(K_1)T_1(K_1 + \mathrm{i}\pi/2) = (2\mathrm{i}\sinh 2K_1)^N \mathbb{1}, \qquad T_2(K_2, H)T_2(-K_2, -H) = \mathbb{1},$$

and thus for $T = T_2^{1/2}T_1T_2^{1/2}$

$$T(K_1, K_2, H)T(K_1 + \mathrm{i}\pi/2, -K_2, -H) = (2\mathrm{i}\sinh 2K_1)^N \mathbb{1}$$

leading to a similar relation for the partition function per site:

$$Z(K_1, K_2, H)Z(K_1 + \mathrm{i}\pi/2, -K_2, -H) = 2\mathrm{i}\sinh 2K_1.$$

This relation can also be verified directly using a partially resummed expansion. Instead of resumming a low-temperature expansion simultaneously in one of the coupling constants and in the magnetic field (Baxter 1980), it is convenient to consider a high-temperature expansion and to resum it in one coupling constant only: in the corresponding diagrammatic expansion, only closed polygons or lines with fields at all ends can appear. Let us introduce the following notations:

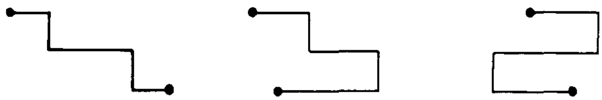
$$t=\tanh K_1, \qquad t'=\tanh K_2, \qquad \tau=\tanh H$$

$$Z(K_1,K_2,H)=\cosh H\, Z(K_1,K_2,0)\Lambda(t,t',\tau).$$

$Z(K_1, K_2, 0)$ is the (known) partition function per site of the two-dimensional model without the field. Λ is thus the correction brought by the magnetic field. Up to the fourth order the expansion of $\ln\Lambda$ is given by the following diagrams:

$\ln\Lambda(t,t',\tau)=$ $\dfrac{t}{1-t}\tau^2$ (0,2)

 $\left(\dfrac{1+t}{1-t}\right)^2 t'\tau^2$ (1, 2)



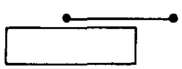
 $\left(\dfrac{1+t}{1-t}\right)^3 t'^2\tau^2$

 $\dfrac{2t}{(1-t)^3}t'^2\tau^2$

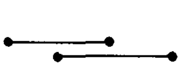
 $\dfrac{4t^3}{(1-t)^2(1-t^2)}t'^2\tau^2$

 $\dfrac{2t^5}{(1-t)(1-t^2)^2}t'^2\tau^2$ (2, 2)

and the disconnected terms:

 $(-2)\sum_{p,q\geqslant 1}(p+q-1)t^{p+2q}t'^2\tau^2$

$$=(-2)\left(\frac{-t^3}{(1-t)(1-t^2)}+\frac{t^3}{(1-t^2)(1-t)^2}+\frac{t^3}{(1-t)(1-t^2)^2}\right)t'^2\tau^2 \qquad (2,2)$$

 $(-\tfrac{1}{2})\sum_{p,q\geqslant 1}(p+q+1)t^{p+q}\tau^4$

$$=(-\tfrac{1}{2})\left(\frac{t^2}{(1-t)^2}+\frac{2t^2}{(1-t)^3}\right)\tau^4 \qquad (0,4).$$

Since the inverse relation is already verified by the partition function without the magnetic field (Baxter 1980), the inverse relation reduces to

$$\ln\Lambda(t,t',\tau)+\ln\Lambda(1/t,-t',-\tau)=\ln(1-\tau^2).$$

It is then easy to verify that the terms of order (0, 2) and (0, 4) add to cancel the singularity at $t=1$ and to give $-\tau^2-\tau^4/2$ and that the other terms, of order (1, 2) and (2, 2), eliminate (to give a result which does not depend on t').

Conversely, supposing the inverse relation, one can try to determine the expansion of the partition function using the inverse relation and the other symmetries of the model:

$$\ln \Lambda(t, t', \tau) = \ln \Lambda(t', t, \tau) = \ln \Lambda(t, t', -\tau).$$

It is quite clear that new assumptions on the nature of the singularities are needed. At the lowest orders, a qualitative study of the diagrams convinces one that only $t = 1$ or $t = -1$ singularities can occur and that the expansion can be written

$$\ln \Lambda(t, t', \tau) = \tau^2 \left[\frac{t}{1-t} + \left(\frac{1+t}{1-t}\right)^2 t' + \left(\frac{\alpha + \beta t + \gamma t^2 + \delta t^3 + \varepsilon t^4 + \eta t^5}{(1-t)^3(1+t)^2}\right) t'^2 + \left(\frac{a' + b't + c't^2 + d't^3 + e't^4 + f't^5 + g't^6 + h't^7}{(1-t)^4(1+t)^3}\right) t'^3 + \dots \right].$$

The inverse relation then implies the following equalities:

$$\alpha = \eta, \beta = \varepsilon, \gamma = \delta; \qquad a' = h', b' = g', c' = f', d' = e'.$$

Hence the order of the singularity at $t = -1$ is reduced from three to two:

$$\left(\frac{a + bt + ct^2 + dt^3 + ct^4 + bt^5 + at^6}{(1-t)^4(1+t)^2}\right) t'^3.$$

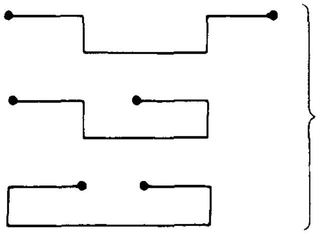
The symmetry between t and t' leads to the following equalities:

$$\alpha = 1, \beta = 7; \qquad a = 1, b = 10; \qquad 2\gamma - 1 = c. \tag{1}$$

At this order, there only exist two undetermined coefficients γ and d. One can notice that the spatial symmetry allows one to exhibit equalities like (1) for any order but that there always remain, at any order, undetermined coefficients in the rational functions. Some information in addition to the knowledge of the t-singularities is then necessary to determine these coefficients. For that purpose it seems natural to recall the Lee–Yang theorem which gives some constraints on the partition function of the Ising model with a magnetic field in any dimension (Lee and Yang 1952). In particular, it is possible to rewrite these constraints on an expansion of the model. Following Bessis *et al* (1976), we introduce $u = e^{-2K} = (1-t)/(1+t)$, $u' = e^{-2K'} = (1-t')/(1+t')$, and the Lee–Yang theorem allows one to write

$$\ln \Lambda(t, t', \tau) = \sum_l [(1-\tau^2)^l - 1] P_l(u, u')$$

where P_l is a polynomial of order $2l$ in u, u'. The inverse relation cannot be directly exploited on this particular expansion: it would involve an infinite number of polynomials. Nonetheless, in the isotropic case, the polynomial P_l is divisible by $(1-u^2)^l$ and hence only contributes at orders greater than $2l$. This polynomial can be seen as the sum of two contributions: a polynomial P_l which is easily computed as it corresponds to the same problem but on a Bethe lattice with the same coordination number ($c = 4$), and a polynomial correction ΔP_l which only occurs for $l \geq 4$ in the case of the square lattice. If one compares this new expansion with the partially resummed expansion we have introduced, in their common domain, that is to say on a high-temperature isotropic expansion, one can see that the coefficients (γ and d) which are not determined by the inverse relation and symmetry relations precisely belong to the unknown corrections to the Bethe lattice contribution. Moreover, it is not possible to restrict the inverse relation to this correction since one can easily check on a resummed diagrammatic expansion that the partition function of the Bethe lattice in a magnetic field does not satisfy the inverse relation. Indeed, this expansion is identical to that of the square lattice at the first two orders in t', and differs from it at the order t'^2 by the diagrams (no disconnected term):



$$\frac{2t}{1-t}\left(\frac{1+t}{1-t}\right)^2 t'^2$$

$$\ln \Lambda_{\text{Bethe}} = \tau^2\left[\frac{t}{1-t}+\left(\frac{1+t}{1-t}\right)^2 t' + \left(\frac{1+5t+7t^2+3t^3}{(1-t)^3}\right)t'^2+\dots\right].$$

Let us note that this case provides an example of a problem for which a local inverse relation does not imply an inverse functional equation on the partition function.

Thus the nature of the analyticity of the partition function with respect to the magnetic field (which is deduced from the Lee–Yang theorem) does not seem to introduce new constraints besides the one coming from the automorphy group generated by the inverse and spatial symmetries. In fact, the magnetic field seems to remain a spectator with respect to the action of the group on the partition function (the same situation holds in higher dimensions or for the Potts model with a magnetic field).

2.3. *The Potts model in three dimensions*

2.3.1. The inverse relation. Because of the preceding difficulties, it seems interesting to study the case of three-dimensional models, since the size of the group increases with the number of parameters characterising the model. The partition function per site Z of the Potts model will be written

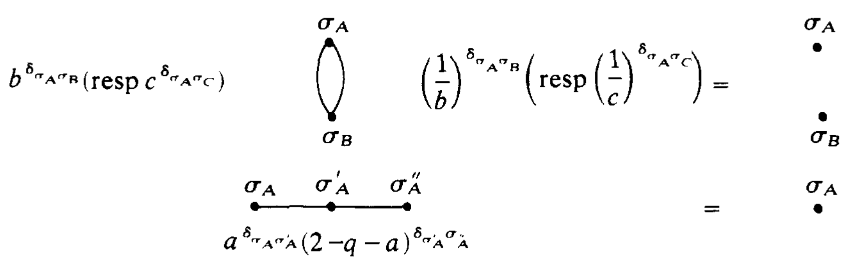
$$Z^{LMN}(a,b,c)=\sum_{\{\sigma\}}\prod_{\langle ij\,\text{hor}\rangle} a^{\delta_{\sigma_i\sigma_j}}\prod_{\langle kl\,\text{hor}\rangle} b^{\delta_{\sigma_k\sigma_l}}\prod_{\langle mn\,\text{ver}\rangle} c^{\delta_{\sigma_m\sigma_n}}, \qquad \sigma\in\mathbb{Z}q.$$

In fact a geometrical derivation allows one to establish the following inverse relation. Let us introduce (see figure 2)

$$T_1 = a^{\delta_{\sigma_A\sigma'_A}}a^{\delta_{\sigma_B\sigma'_B}}a^{\delta_{\sigma'_C\sigma_C}}\dots$$

$$T_2 = b^{\delta_{\sigma_A\sigma_B}}b^{\delta_{\sigma_B\sigma_D}}b^{\delta_{\sigma_C\sigma_E}}\dots c^{\delta_{\sigma_A\sigma_C}}c^{\delta_{\sigma_C\sigma_F}}\dots\delta_{\sigma_A\sigma'_A}\delta_{\sigma_B\sigma'_B}\dots.$$

The local relations which can be represented by



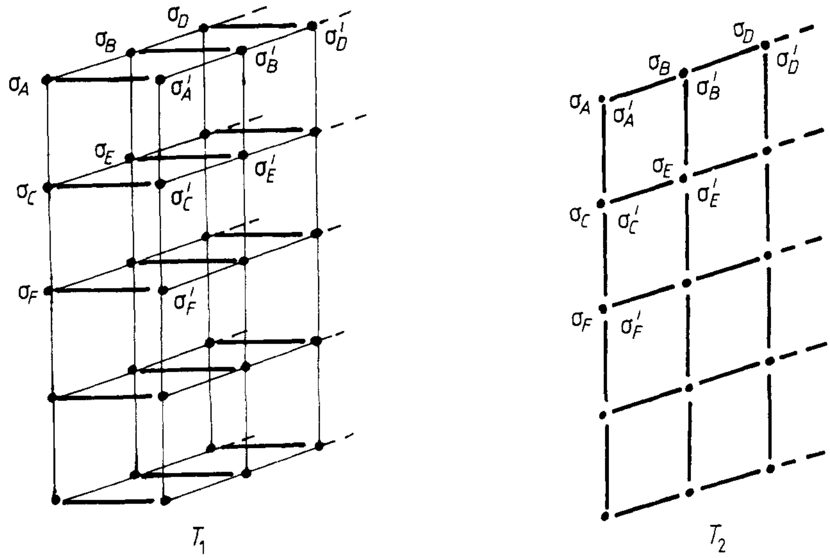


Figure 2.

imply the following matrix relations:

$$T_1(a)T_1(2-q-a)=(a-1)^{LN}(1-q-a)^{LN}\mathbb{1}, \qquad T_2(b,c)T_2(1/b,1/c)=\mathbb{1}.$$

$T=T_2^{1/2}T_1T_2^{1/2}$ then satisfies an inverse matrix relation, with its corresponding inverse relation for the partition function per site :

$$Z(a,b,c)Z(2-q-a,1/b,1/c)=(a-1)(1-q-a).$$

2.3.2. The automorphy group. The inverse relation can be combined with the spatial symmetries of the model, $Z(a,b,c)=Z(\sigma(a,b,c))$ (where σ designates any element of the permutation group S_3), to generate an infinite discrete automorphy group for the partition function per site. To elucidate the structure of this group G, it is convenient to introduce the following canonical variables:

$$x=\frac{a-q_+}{a-q_-}, \qquad y=\frac{b-q_+}{b-q_-}, \qquad z=\frac{c-q_+}{c-q_-}; \qquad q_\pm^2-(2-q)q_\pm+1=0.$$

With these new variables the action of the group can be written:

$$I:(x,y,z)\mapsto(1/x,q_+^2/y,q_+^2/z), \qquad S:(x,y,z)\mapsto(y,x,z),$$
$$\sigma:(x,y,z)\mapsto\sigma(x,y,z), \qquad S':(x,y,z)\mapsto(z,y,x),$$

where σ is a general element of S_3 and S, S' are two generating transpositions of S_3. Every element g of the group G can be decomposed as $g=\sigma_0I\sigma_1I\sigma_2\ldots I\sigma_n$. One can thus designate the following subgroups:

$$G_0=\{g;n\text{ even}\}, \qquad G_1=\{g;\sigma_0,\sigma_1\ldots\sigma_n=e\},$$

(where e is the identity element of G), $H=G_0\cap G_1$. One can verify that the elements of H correspond to simple dilatations of the three variables x, y, z that preserve the product xyz. Noting that the third transposition, $S'' = SS'S = S'SS'$, commutes with I, one can easily see that H is generated by the elements of the form

$$\sigma I \sigma' I (\sigma\sigma')^{-1}, \qquad \sigma, \sigma' = e, S, S', SS', S'S,$$

and more precisely by the two remarkable elements

$$(SI)^2 : (x, y, z) \mapsto (q_+^2 x, y/q_+^2, z), \qquad (S'I)^2 : (x, y, z) \mapsto (q_+^2 x, y, z/q_+^2),$$

which commute: H is thus isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Moreover, H is normal in each of the subgroups G_0 and G_1, which are themselves normal in G, these properties being translated by the following exact sequences:

$$0 \to H \to G_1 \to \mathbb{Z}_2 \to 0, \qquad 0 \to G_1 \to G \to S_3 \to 0,$$

or

$$0 \to H \to G_0 \to S_3 \to 0, \qquad 0 \to G_0 \to G \to \mathbb{Z}_2 \to 0.$$

As the elements of the group can be represented by automorphic functional equations on the partition function, the images of a singular point under the action of the group are also singular points. Thus the singular (or critical) manifolds have to be looked for among the manifolds which are stable under the action of the group. Of course, this property is not sufficient to determine the critical manifolds. A natural supplementary assumption (which is a straightforward generalisation of the situation occurring in two dimensions) is that the manifold be algebraic in the initial parameters a, b, c. Considering the stability under the action of the normal subgroup H, it is easily seen that the only algebraic invariant is xyz, and that the equation for the critical manifold must be $xyz = C$. The stability under the action of the inverse I then determines the constant $C: C = \pm q_+^2$. Unfortunately such an equation leads, in the isotropic case, to a value for the critical temperature (depending on q) which is definitely different from the numerical data (Blöte and Swendsen 1979, Ditzian and Kadanoff 1979). Hence, one must conclude that the critical manifold of the three-dimensional Potts model cannot be algebraic in the initial parameters of the model, a conclusion which also holds in higher dimensions. Let us also remark that, as in the two-dimensional case, the subgroup H becomes finite for some remarkable values of $q: q = 2 + 2\cos 2\pi m/n$ where m and n are integers (for $m = 1$ these are the Beraha numbers (Beraha *et al* 1980)).

2.3.3. Diagrammatic expansion. As before, one can use a high-temperature partially resummed diagrammatic expansion and verify explicitly, up to a certain order, the inverse functional relation on the partition function. Extracting a leading high-temperature term

$$Z(a, b, c) = (a + q - 1)(b + q - 1)(c + q - 1)q^{-3/2}\Lambda(a, b, c)$$

and making the change of variables

$$u = \frac{a-1}{a+q-1}, \qquad v = \frac{b-1}{b+q-1}, \qquad w = \frac{c-1}{c+q-1},$$

the inverse relation can be rewritten

$$\ln \Lambda(u, v, w) + \ln \Lambda\left(\frac{1}{u}, \frac{-v}{1+(q-2)v}, \frac{-w}{1+(q-2)w}\right)$$
$$= \ln \frac{(1-v)[1+(q-1)v](1-w)[1+(q-1)w]}{(1+(q-2)v)(1+(q-2)w)}$$

where $\ln \Lambda$ has the following expansion:

$$\ln \Lambda(u, v, w) = f_2(u)v^2 + f_2'(u)w^2 + f_3(u)v^3 + f_3'(u)w^3 + f_4(u)v^4 + g(u)v^2w^2 + f_4'(u)w^4 + \dots$$

where $f_i(u)$, $f_i'(u)$ and $g(u)$ are rational functions of the variable u, which can be obtained by summing up all the diagrams, of a given order in v and w, in the direction associated to u. One recognises with the functions $f_i(u)$ and $f_i'(u)$ the first terms of the resummed expansion of the two-dimensional partition function of the Potts model. Since this last model satisfies the inverse relation with the same automorphic factor

$$\frac{(1-v)[1+(q-1)v]}{1+(q-2)v} \qquad \left(\text{resp } \frac{(1-w)[1+(q-1)w]}{1+(q-2)w}\right)$$

the inverse relation, up to the fourth order in (v, w), reduces to $g(u)+g(1/u)=0$. The function g has resummed diagrams of two distinct types: on the one hand, diagrams which are identical to those of the Ising model (Jaekel and Maillard 1982a) but now with an extra $(q-1)$ coefficient for the connected diagrams (contribution $g_c(u)$) and a $(q-1)^2$ coefficient for the disconnected diagrams (contribution $g_d(u)$); on the other hand, diagrams which are specific to the Potts model and which appear with a $(q-1)(q-2)$ coefficient (contribution $g_P(u)$) (see figure 3). The inverse relation on the Ising model gives the equation

$$g_d(u) + g_c(u) + g_d(1/u) + g_c(1/u) = 0.$$

Therefore, to verify the inverse relation on the Potts model one needs to prove the

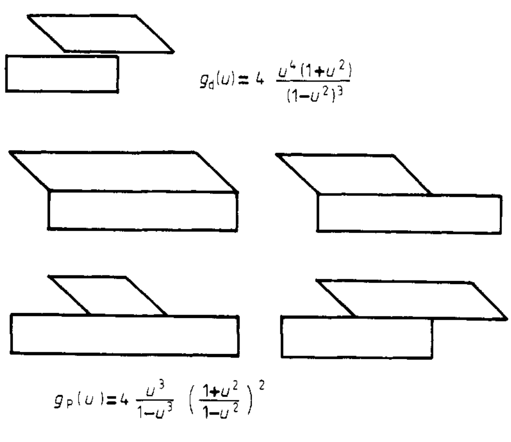


Figure 3.

equation

$$(q-1)[g_c(u)+g_c(1/u)]+(q-1)^2[g_d(u)+g_d(1/u)]+(q-1)(q-2)[g_P(u)+g_P(1/u)]=0$$
$$\Leftrightarrow (q-1)(q-2)[g_d(u)+g_d(1/u)+g_P(u)+g_P(1/u)]=0. \qquad (2)$$

The exact expressions of g_d and g_P which are given in figure 3 show that the singularity at $u^3=1$ cancels and satisfies equation (2), proving the exactness of the inverse relation for the three-dimensional Potts model up to the fourth order.

Reciprocally, considering the coefficients of the expansion as rational functions (in the variable u) whose singularities are deduced from the general shape of the diagrams, one can use the inverse and symmetry functional relations to determine the exact expressions of these rational functions. The difficulties that occurred in the two-dimensional case also occur here: among other ones the presence, at the $2r$th order, of singularities for all the nth roots of unity (with $n \leq r+1$) makes it impossible to cancel all the arbitrariness in the determination of the rational functions. This also raises the question of the type of singularities which occur for the three-dimensional Ising model with more acuity. For instance, the diagram of figure 4 suggests a $u^4=1$ singularity. However, the same kind of diagrams are present for the two-dimensional Ising model (figure 5), although we know, from the exact solution for this model, that only $u^2=1$ singularities occur.

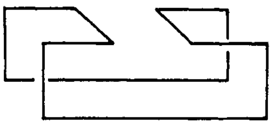


Figure 4.

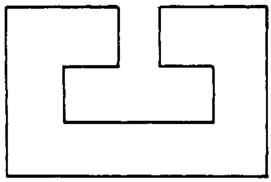


Figure 5.

Let us note that the partition functions of the simple cubic and triangular Potts models satisfy the same inverse and symmetry functional equations. The preceding geometrical derivation and resummed diagrammatic approach are easily applied to the triangular lattice leading, with the same notations and definitions, to the following expansion:

$$\ln \Lambda(u,v,w)=(q-1)\frac{u^2}{1-u^2}(v^2+w^2)+2(q-1)\frac{u}{1-u^2}vw+\ldots.$$

Even in the Ising case the coefficient of the uvw term in the triangular case (just as the coefficient of the $u^2v^2w^2$ term for the simple cubic lattice) cannot be determined by the constraints that the inverse and symmetry relations imply. These models then raise the question of the type of information one must add to the analyticity criteria and to the inverse and symmetry relations in order to determine the partition function and to distinguish between different solutions of the same set of functional equations. Let us remark that, for instance, another property which is deeply linked to their common automorphy properties distinguishes between these two models: the equation $xyz=q_+^2$ (which can be deduced from the group structure) identifies with the conjecture given for the critical variety of the triangular lattice (Hintermann *et al* 1981), although the critical manifold for the simple cubic lattice is different from this equation, as was seen before.

3. Correlation functions

Up to now the inverse relation has been studied on transfer matrices or partition functions which correspond to their largest eigenvalue. However, this relation seems to be satisfied by the other eigenvalues of the transfer matrix, and it is thus natural to look for an inverse relation on the correlation functions.

3.1. Geometrical derivation

The same geometrical derivation which has been used to deduce the inverse functional relation on the transfer matrix can be generalised to the correlation functions.

3.1.1. Two-point correlation functions. In order to make explicit the approach, using a simple example, we shall first restrict ourselves to the two-point correlation functions of the two-dimensional Ising model. The latter will be written ($-\beta H = \Sigma_{i,j,k,l} K_1\sigma_i\sigma_j + K_2\sigma_k\sigma_l$)

$$\langle\sigma_i\sigma_j\rangle = \frac{\Sigma_{\{\sigma\}}\sigma_i\sigma_j\,\mathrm{e}^{-\beta H}}{\Sigma_{\{\sigma\}}\mathrm{e}^{-\beta H}} = g(K_1, K_2)$$

where $j = i + r$, $r = (m, n)$, $m, n \in \mathbb{Z}$.

It will appear in the following that one must distinguish two different kinds of correlation functions for $m \leqslant n$ and for $m > n$, and these will be respectively represented by the two examples $(m, n) = (2, 3)$ and $(m, n) = (3, 2)$ (figures 6(*a*), 7(*a*)).

A first step is to show an appropriate inverse local relation. For this purpose we introduce the following elementary cell: the smallest rectangular cell containing the two points i and j, with the usual anisotropic coupling constants of the square lattice (K_1 and K_2) and an additional coupling constant K which links the two points i and j (see figures 6(*a*), 7(*a*)). Consider next the elementary cell which can be deduced from the preceding one by a symmetry with respect to the vertical line and by the following change on the coupling constants: $K_1 \to K_1 + \mathrm{i}\pi/2$, $K_2 \to -K_2$, $K \to -K$. Let us bind the two cells as indicated in figures 6(*b*), 7(*b*). The coupling constants K_2 and $-K_2$ along the vertical line disappear, thus leading to the equivalent figures 6, 7(*c*): the spins $\sigma_1, \sigma_2, \ldots$ become independent, making it now possible to sum over them. The consequence of this summation is to identify the spins $\sigma'_1, \sigma''_1, \ldots$ and respectively $\sigma'_2, \sigma''_2, \ldots$. This last equivalence is represented by figures 6(*d*), 7(*d*). In the $m \leqslant n$ case these two operations can be repeated (see figures 6(*e*)–(*f*)) until one finally obtains figure 6(*f*), which means that the following spins are identified: $\tilde{\sigma}_1 = \ldots = \tilde{\tilde{\sigma}}_1$ and $\tilde{\sigma}_2 = \ldots = \tilde{\tilde{\sigma}}_2$. On the contrary, in the $m > n$ case one can see that these two operations lead to figure 7(*e*) from which it is impossible to obtain the required identification of the spins. This phenomenon, here exhibited in the (3, 2) and (2, 3) cases, is in fact quite general: in order to obtain the required result, that is to say the cancellation of the coupling constants K and $-K$, it is necessary to identify the spins $\tilde{\sigma}_1$ and $\tilde{\tilde{\sigma}}_1$. A summation on the independent spins along a line makes the spins $\tilde{\sigma}_1$ and $\tilde{\tilde{\sigma}}_1$ closer by one unit ($m \to m - 1$) and, at the same time, creates a spin (like the spin σ_2 in figures 6, 7(*e*)) which one can no longer sum over (since it is related to other spins). This obstruction propagates by one unit at each step and the competition between these two phenomena leads to the identification $\tilde{\sigma}_1 = \tilde{\tilde{\sigma}}_1$ only when $m \leqslant n$.

A second step is to show a global inverse relation on an appropriate transfer matrix: the transfer matrix obtained by replicating the preceding elementary cell along

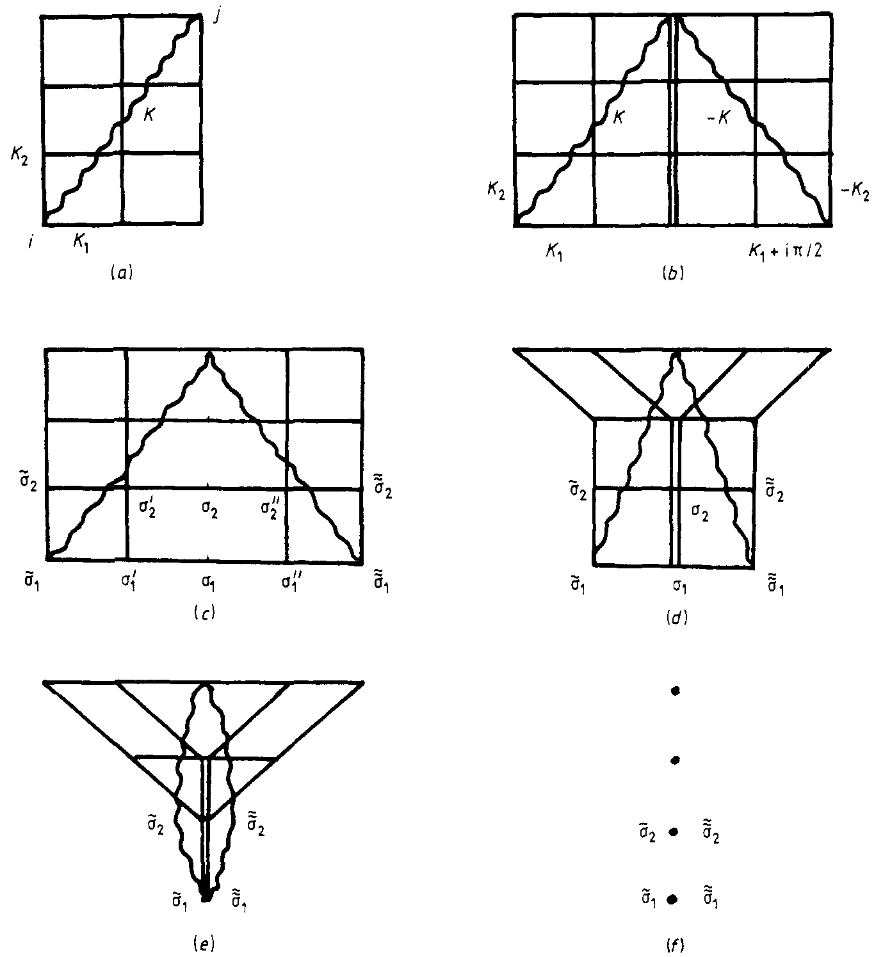


Figure 6.

a vertical line. Then, consider the product of this transfer matrix by the transfer matrix which can be deduced from the preceding one by a reflection with respect to the vertical line and by the inverse transformation on the coupling constants: $K_1 \to K_1 + i\pi/2$, $K_2 \to -K_2$, $K \to -K$. It is easily seen that the iteration of local relations leads to the following result:

$$T(K_1, K_2, K)T(K_1 + i\pi/2, -K_2, -K) = (2i \sinh 2K_1)^N \mathbb{1}.$$

In a last step one can consider the inverse functional relation associated to the partition function per site generated by this transfer matrix, which corresponds to an Ising model on a non-planar lattice:

$$Z(K_1, K_2, K)Z(K_1 + i\pi/2, -K_2, -K) = 2i \sinh 2K_1.$$

Taking the logarithmic derivative with respect to K, at $K = 0$ of the last equation, one obtains the inverse relation for the two-point correlation function: $g(K_1, K_2) = g(K_1 + i\pi/2, -K_2)$.

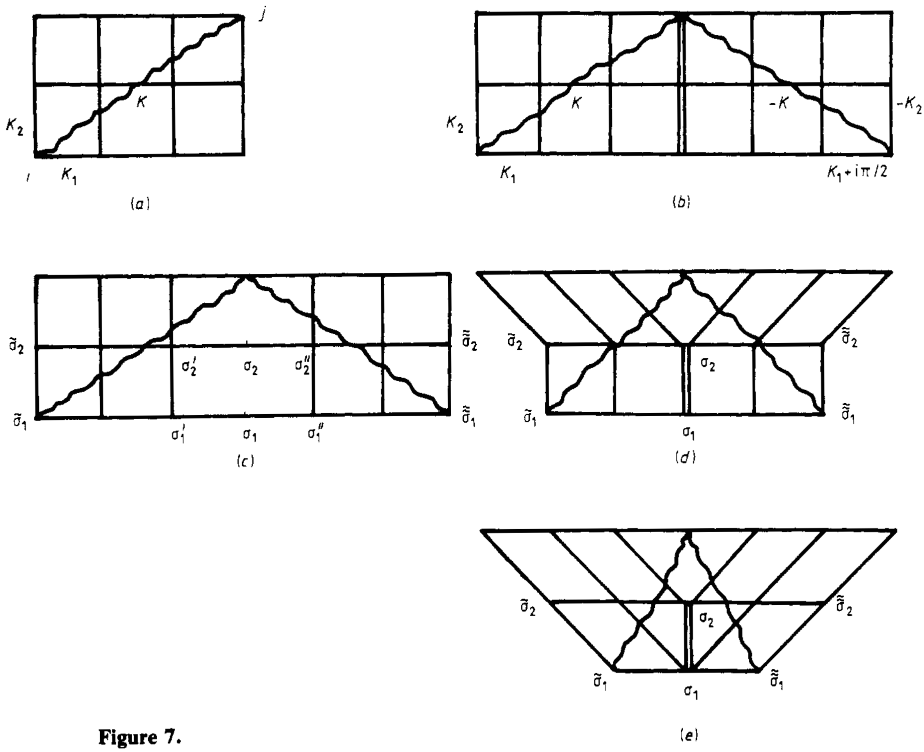


Figure 7.

3.1.2. N-point correlation functions. As before we consider the following example which will represent a general n-point correlation function:

$$\langle \sigma_i \sigma_j \sigma_k \rangle = g_3(K_1, K_2)$$

where $i = j + r$, $r = (m, n)(=(1, 1))$, $j = k + r'$, $r' = (m', n')(=(1, 2))$. Introducing the smallest rectangular cell containing the three points i, j, k with the usual anisotropic coupling constants of the square lattice (K_1, K_2) and with a three-spin interaction between the sites i, j, k (K) (see figure 8), a similar sequence of transformations leads to the following inverse relation on the three-point correlation function: $g_3(K_1 + \mathrm{i}\pi/2, -K_2) = g_3(K_1, K_2)$. Extending the argument which was used for the two-point functions, this is seen to occur only when $m \leqslant n$, $m' \leqslant n'$. One can easily

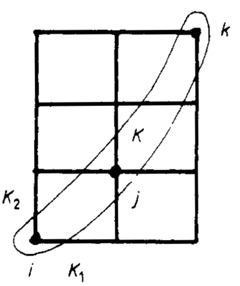


Figure 8.

be convinced that the n-point functions satisfy

$$\langle \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_n} \rangle = g_n(K_1, K_2), \qquad i_{\alpha+1} = i_\alpha + r_\alpha, \qquad r_\alpha = (m_\alpha, n_\alpha),$$

(if $m_\alpha \leqslant n_\alpha$)

$$g_n(K_1, K_2) = g_n(K_1 + i\pi/2, -K_2).$$

3.1.3. Three-dimensional correlation functions. The same approach can be generalised to the simple cubic lattice *mutatis mutandis*. Let us consider (for simplicity see figure 9(a)) the two-point correlation function $\langle \sigma_i \sigma_j \rangle = g_2(K_1, K_2, K_3)$, $j = i + r$, $r = (m, n, p)$ $(=(2, 1, 1))$, where K_1, K_2, K_3 are the three coupling constants of the model. In the general case the following inverse relation is deduced:

$$g_2(K_1, K_2, K_3) = g_2(K_1 + i\pi/2, -K_2, -K_3) \qquad \text{if } m \leqslant n + p.$$

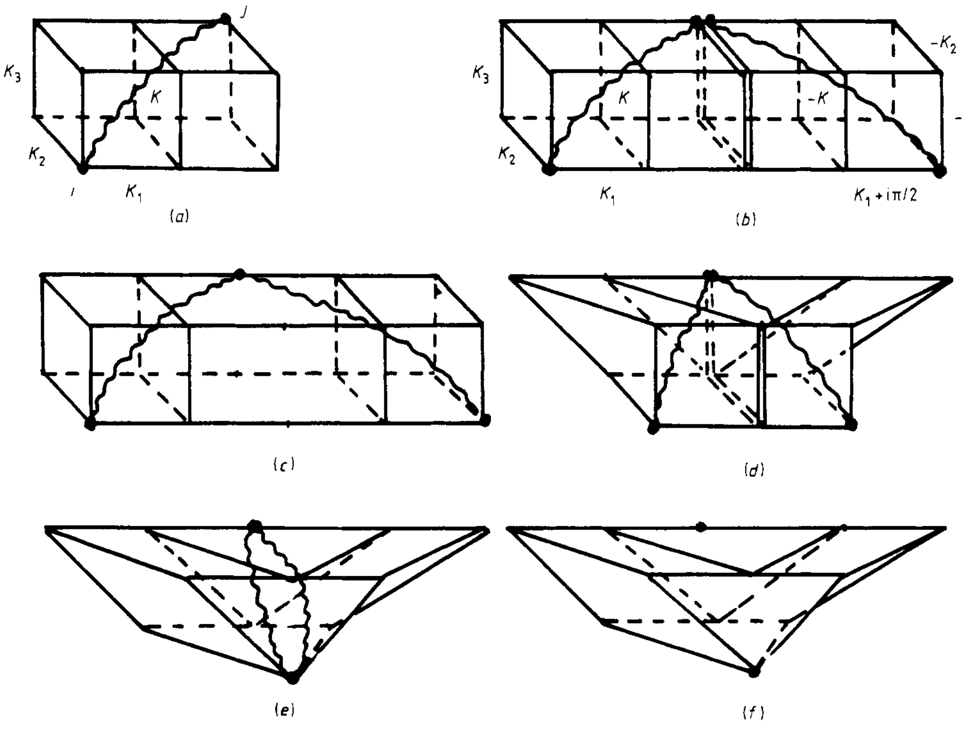


Figure 9.

3.2. *Diagrammatic expansion*

The geometrical approach can only be considered as a heuristic argument, and in order to be convinced of the pertinence of the inverse relation for the correlation functions it is useful to verify it on the first terms of a resummed diagrammatic expansion. This expansion is of the same type as the one which was used for the partition function (with a partial resummation in only one variable), and the exact rules to obtain this high-temperature expansion can be found in Domb (1960). The distinction that exists between $m \leqslant n$ and $m > n$ correlation functions can be exhibited in a simple way by looking at the inverse relation for the two-point correlation functions when the two points are nearest neighbours. Taking the logarithmic derivative of the inverse relation of the partition function with respect to the two coupling constants K_1 and K_2, one obtains the two functional equations:

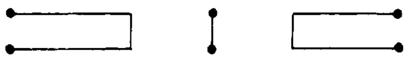
$$g(K_1, K_2) = g(K_1 + \mathrm{i}\pi/2, -K_2), \qquad (3a)$$

$$g'(K_1, K_2) + g'(K_1 + \mathrm{i}\pi/2, -K_2) = 2 \coth 2K_1, \qquad (3b)$$

where g is the (0, 1) correlation function (the two points being on a vertical line), and g' is the (1,0) correlation function (the two points being on a horizontal line). Introducing the notations $t = \tanh K_1$, $t' = \tanh K_2$, these two functions have the following expansions in t':

$$g(t, t') = \frac{1+t^2}{1-t^2} t' + \ldots, \qquad g'(t, t') = t + \frac{2t}{1-t^2} t'^2 + \ldots.$$

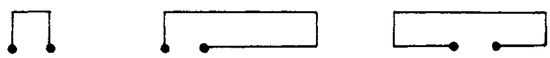
These expansions can be obtained either by differentiating the expansion of the partition function, or by considering the following diagrams:



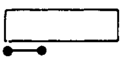
$$t' + 2\frac{t^2}{1-t^2} t'$$

and

 t,



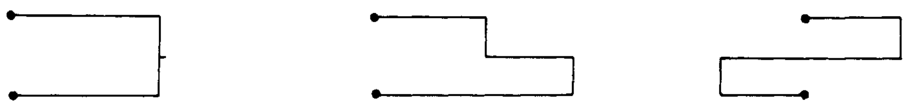
$$\frac{2t}{(1-t^2)^2} t'^2$$



$$(-2)\left(\sum_{p \geqslant 1} p t^{2p+1}\right) t'^2 = -2t \frac{t^2}{(1-t^2)^2} t'^2.$$

One then easily verifies equations $(3a, b)$ at the lowest orders in t' ($t \to 1/t$, $t' \to -t'$).

The vertical two-point correlation functions $(0, n)$ can also be shown to satisfy the inverse relation. For instance, the $(0, 2)$ correlation function has the following expansion:



$$g(t, t') = \frac{1 + 4t^2 + t^4}{(1-t^2)^2} t'^2 + \ldots.$$

An interesting particular case is that of the diagonal two-point correlation functions (n, n):

$$g(t, t') = \binom{2n}{n} \frac{t^n}{(1-t^2)^n} t'^n + \ldots.$$

An inspection of the diagrams shows that the most severe singularity which appears at the lowest order is $(1-t^2)^n$. As terms like $t^p t'^n$ for $p < n$ do not exist, the inverse relation then determines this first term up to a constant factor. The diagonal two-point correlation functions show an additional interesting property: invariance under the spatial symmetry ($t \leftrightarrow t'$). They are therefore invariant under the action of the whole group G generated by the inverse and the symmetry relations. It is then natural to envisage determining the coefficients of the resummed diagrammatic expansion, using the inverse and symmetry relations, and some analyticity hypothesis. Examination of the diagrams at lowest orders indicates a singularity at $t^2 = 1$, and then the inverse and symmetry relations force the correlation functions to be functions of the algebraic invariant

$$k = 4tt'/(1-t^2)(1-t'^2) = \sinh 2K_1 \sinh 2K_2.$$

But, contrary to the case of the partition function, these two relations still leave many coefficients arbitrary, which can be seen to correspond to the expansion of a function of the variable k. These properties can be checked on the exact form of the diagonal two-point correlation functions (n, n), when expressed as Toeplitz determinants with coefficients depending only on the variable k (MacCoy and Wu 1973).

It is clear that the geometrical approach and the diagrammatic expansion apply indifferently to Potts models with or without field, on various regular lattices. Their n-point correlation functions satisfy the same functional equation with the same restriction on the positions of the points ($m \leq n$, $m \leq n+p, \ldots$). However, the diagonal correlation functions will no longer depend only on the algebraic invariant of the group G ($xy/-q_+$ or $(bc-1)/(b+c+q-2)$, for the square lattice) (Rammal and Maillard 1983). Let us remark in that context that the square of the magnetisation can be seen as the limit of a diagonal two-point correlation function whose points are separated by a distance growing to infinity (for $T < T_c$). In the Ising case, it can be seen to depend only on the invariant k ($M = (1-k^{-2})^{1/8}$, which is consistent with the previous results.

It should be noticed that the functional equations which were obtained for the correlation functions do not involve at the same time functions with different numbers of points. They hold for a unique correlation function, the n points being fixed and the coupling constants being varied. They are thus of the same type as those which were derived by Fisher (1959), or Baxter and Enting (1978), for the nearest-neighbour two-point functions of the hexagonal lattice. They are to be distinguished from equations relating different n-point functions (Groeneveld *et al* 1978, Kasteleyn and Boel 1979), and from differential equations with respect to the positions of the points (MacCoy and Wu 1973, Sato *et al* 1978–1980).

4. Conclusion

The inverse and symmetry relations (and the group generated by these two transformations) have been shown to constrain the partition functions of various models. Such constraints seem to be less severe than the ones resulting from a generalised star-triangle or tetrahedron relation. On the other hand they can be exhibited on most well known models such as Potts models in any dimension (with field), or vertex models (non-symmetric eight or sixteen vertex models). These relations are also more constraining than a single self-duality relation, and appear in models that are not self-dual (three-dimensional models).

The general character of these functional relations also deserves attention. For instance, quite similar relations make their appearance in familiar problems such as the harmonic and anharmonic oscillators, or else, ordinary differential equations with irregular singularities (Sibuya and Cameron 1972): $y'' - (x^3 + \lambda)y = 0$, the Stokes multipliers satisfying the functional equation

$$f(\lambda) + f(\omega\lambda)f(\omega^{-1}\lambda) = 1, \qquad \omega^5 = 1.$$

The same functional equation is also satisfied by the partition function of the hard-hexagon model (Baxter and Pearce 1982). In all cases, the resolution of the functional equation will require additional analyticity hypotheses or boundary conditions. In a parallel approach, the quantum spectral transform (Izergin and Korepin 1981) also puts into evidence inverse functional equations which are there related to the notion of quantum determinants.

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